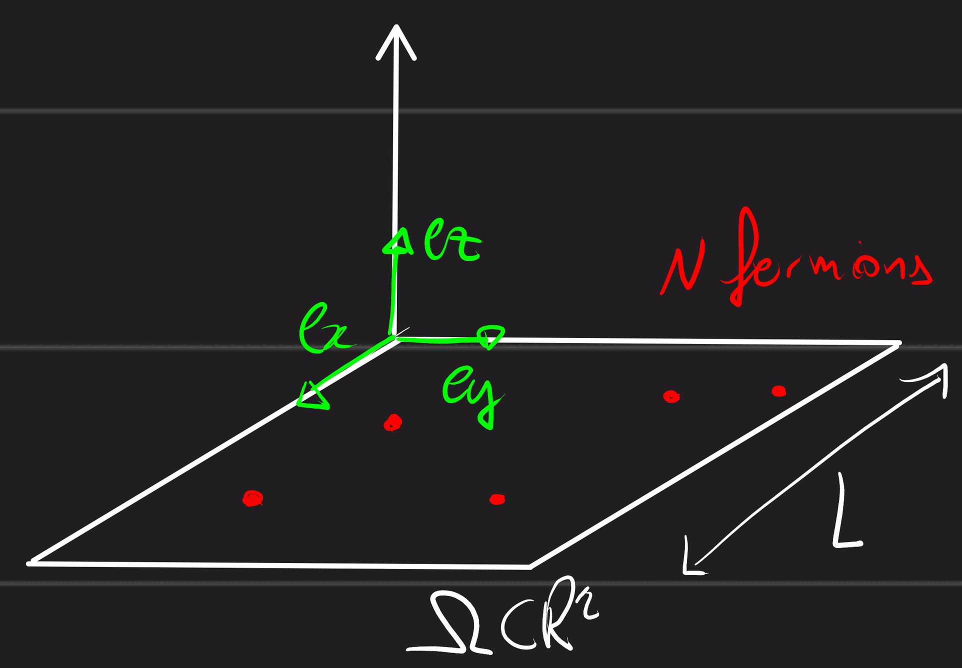
Multiple Landau level filling for a large magnetic field limit of 2d fermions

Model:

context:



$$\begin{cases} \int_{\Omega^N} |\Psi_N(x_{1:N})|^2 \, dx_{1:N} = 1 \\ \sigma \in S_N, \quad \Psi_N(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varepsilon(\sigma)\, \Psi_N(x_{1:N}) \quad \circledast \end{cases}$$

$$\Psi_N \in L^2_-(\Omega^N) := \{ \Psi \in L^2(\Omega^N) \mid \forall \sigma \in S_N \ \circledast \}$$

$$L^2_-(\Omega^N) := \bigwedge_{i=1}^{N} L^2(\Omega) = \overline{\operatorname{span}}\{ \phi_1 \wedge \dots \wedge \phi_N,\ \phi_1, \dots, \phi_N \in L^2(\Omega) \}$$

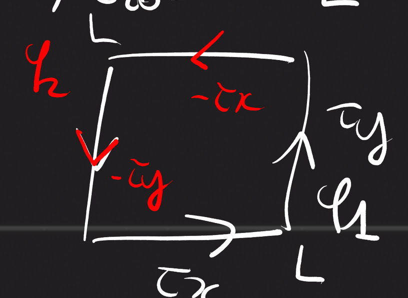
Pauli principle: $\Psi \in L^2(\Omega)$, $\Psi \wedge \Psi = 0$

1 body operator: $\mathcal{L} := (\underbrace{-i\hbar\nabla}_{L^{-1}} - \underbrace{b}_{L} A)^2$ $(m=1, c=1, q=1)$, $\frac{\hbar}{b} =: \ell_b^2$, $\nabla \wedge A = e_z$, $A = \nabla^\perp \phi = \begin{pmatrix} -\partial_y \phi \\ \partial_x \phi \end{pmatrix}$ (smooth), ex: $A_{Lan} := \begin{pmatrix} -y \\ 0 \end{pmatrix}$ $(\nabla \cdot A = 0)$

• magnetic periodic boundary conditions: ⚠ $P := (i\hbar\nabla + bA)$, $T_{z_0}\Psi := \Psi(\cdot - z_0)$, $[P, T_{z_0}] = b[A, T_{z_0}] = b(A - T_{z_0}A)T_{z_0} \neq 0$

→ Translation symmetry broken by magnetic field. $\nabla \wedge b(A - T_{z_0}A) = 0 \Rightarrow b(A - T_{z_0}A) = \hbar \nabla \varphi_{z_0}$,

$$\tau_{z_0} := e^{i\varphi_{z_0}} T_{z_0}, \quad [P, \tau_{z_0}] = [i\hbar\nabla + bA, e^{i\varphi_{z_0}} T_{z_0}] = \left(-\hbar\nabla\varphi_{z_0} + b(A - T_{z_0}A)\right)\tau_{z_0} = 0$$



$$[\tau_x, \tau_y] = \left(e^{i\varphi_1} - e^{i\varphi_2}\right) T_x T_y \qquad \varphi_1 - \varphi_2 = \frac{1}{\ell_b^2} \int_{\partial\Omega} A \cdot dl = \frac{1}{\ell_b^2} \int_\Omega B \cdot dS = \frac{L^2}{\ell_b^2} \text{ so } [\tau_x, \tau_y] = 0 \iff \frac{L^2}{\ell_b^2} = 2\pi d$$

$$H^k_{mp}(\Omega) := \{ \Psi \in H^k(\Omega) \mid \exists \tilde{\Psi} : \mathbb{R}^2 \to \mathbb{C} \mid \tilde{\Psi}_{|\Omega} = \Psi,\ \tau_x \tilde{\Psi} = \tilde{\Psi},\ \tau_y \tilde{\Psi} = \tilde{\Psi} \} \qquad \mathcal{D}om(\mathcal{L}) := H^2_{mp}(\Omega)$$

• Landau levels: $\mathcal{L} = \sum_{n \in \mathbb{N}} E_n \Pi_n$, $n LL := \Pi_n H^2_{mp}(\Omega)$ ↳ rank d projector, $E_n := 2\hbar b(n + \frac{1}{2})$, $\sum_{n \in \mathbb{N}} \Pi_n = 1$

N body operator: $$\mathcal{H}_N := \underbrace{\sum_{i=1}^{N} \left(\mathcal{L}(x_i) + V(x_i) \right)}_{N} + \frac{2}{N-1} \underbrace{\sum_{i<j} w_{ij}(x_i - x_j)}_{N \cdot \frac{N-1}{2}}, \quad \mathcal{D}om(\mathcal{H}_N) := \bigwedge_{i=1}^{N} \mathcal{D}om(\mathcal{L})$$

ground state energy: $$E^0_N := \min_{\substack{\Psi_N \in \mathcal{D}om(\mathcal{H}_N) \\ \|\Psi_N\| = 1}} \langle \Psi_N | \mathcal{H}_N \Psi_N \rangle, \quad V, w \in L^2(\Omega)$$

Scaling: density: $\frac{L^2}{N}$, $\begin{cases} \frac{L}{\sqrt{N}} : \text{mean distance between particles} \\ \ell_b : \text{minimal distance between particles in a fixed LL} \end{cases}$ $\left(\frac{L/\sqrt{N}}{\ell_b} \right)^2 = \frac{bL^2}{N\hbar}$

Lieb-Solovej-Yngvason '95: in $\mathbb{R}^2$ with $w(x-y) = \frac{1}{|x-y|}$

• $\frac{L^2}{N\ell_b^2} \to +\infty$ all particles in 0LL → electrostatic model $\rho \mapsto \int V\rho + \int w \rho^{\otimes 2} = \int (V + w * \rho)\rho$

• $\frac{L^2}{N\ell_b^2} \to 0$ all LL filled : $\int \rho^{1 + d/2} + \int (V + w * \rho)\rho$

Fournais Lewin Solovej '15, $\mathbb{R}^d$; Fournais Madsen '19, $\mathbb{R}^3$ } magnetic TF with general V, w: $\int \mathcal{J}(\rho) + \int (V + w * \rho)\rho$ ↳ depends on the scaling

Quantum Hall effect: $0LL, \dots, (q-1)LL$ filled, qLL partially filled with ratio r,

$2\pi d = \frac{L^2}{\ell_b^2}$, fix $\frac{N}{d} \to \underbrace{q}_{\in \mathbb{N}} + \underbrace{r}_{\in(0,1)}$ so $\frac{L^2}{N\ell_b^2} = 2\pi \frac{d}{N} \to \frac{2\pi}{(q+r)}$, d finite $\Leftrightarrow$ Ω bounded, semi classical: $\hbar = N^{-\delta}$

$\ell_b = O(N^{-1/2}) \Rightarrow b = \frac{\hbar}{\ell_b^2} = O(N^{1-\delta}) \Rightarrow$ kinetic energy : $\hbar b = O(N^{1-2\delta}) \xrightarrow{} +\infty \to \frac{1}{4} < \delta < \frac{1}{2}$

<u>Electrostatic model for qLL:</u> $\mathcal{D}_{qLL} := \left\{ \rho \in L^1(\Omega) \,\middle|\, \int_\Omega \rho \, dx = \overbrace{\frac{r}{q+r}}^{\text{proportion of particles in } qLL},\ 0 \leq \rho \leq \underbrace{\frac{1}{L^2(q+r)}}_{\text{Pauli principle}} \right\}$

$\underset{\in \mathcal{D}_{qLL}}{\mathcal{E}_{qLL}[\rho]} := \int_\Omega V(x)\rho(x)\,dx + \int_{\Omega^2} \omega(x-y)\rho(x)\rho(y)\,dx\,dy$, $E^\circ_{qLL} := \min_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}$

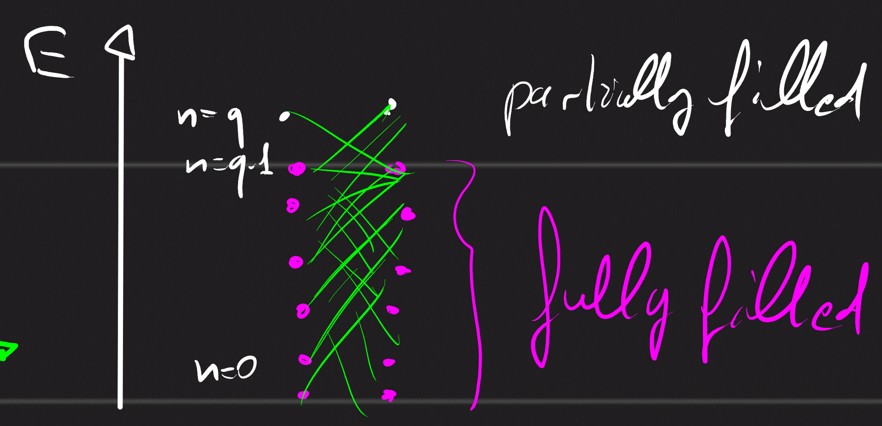
<u>Results:</u>

Theorem (Convergence of the ground state energy)

$$\boxed{\text{If } V, \omega \in L^2(\Omega), \text{ then } \quad \frac{E^\circ_N}{N} = \hbar b\, E^{q,r} + E^{q,r}_V + E^{q,r}_\omega + E^\circ_{qLL} + o(1)}$$

If the system as $0LL, \dots, (q-1)LL$ filled, qLL partially filled with ratio r then

- $\underbrace{\hbar b\, E^{q,r}}_{\text{order of kinetic term}}$: kinetic energy
- $E^{q,r}_V$: potential energy of the q lowest LL
- $E^{q,r}_\omega$: interactions between q lowest LL and interactions between q lowest LL and qLL
- E°_{qLL} : potential energy of qLL and interaction inside qLL



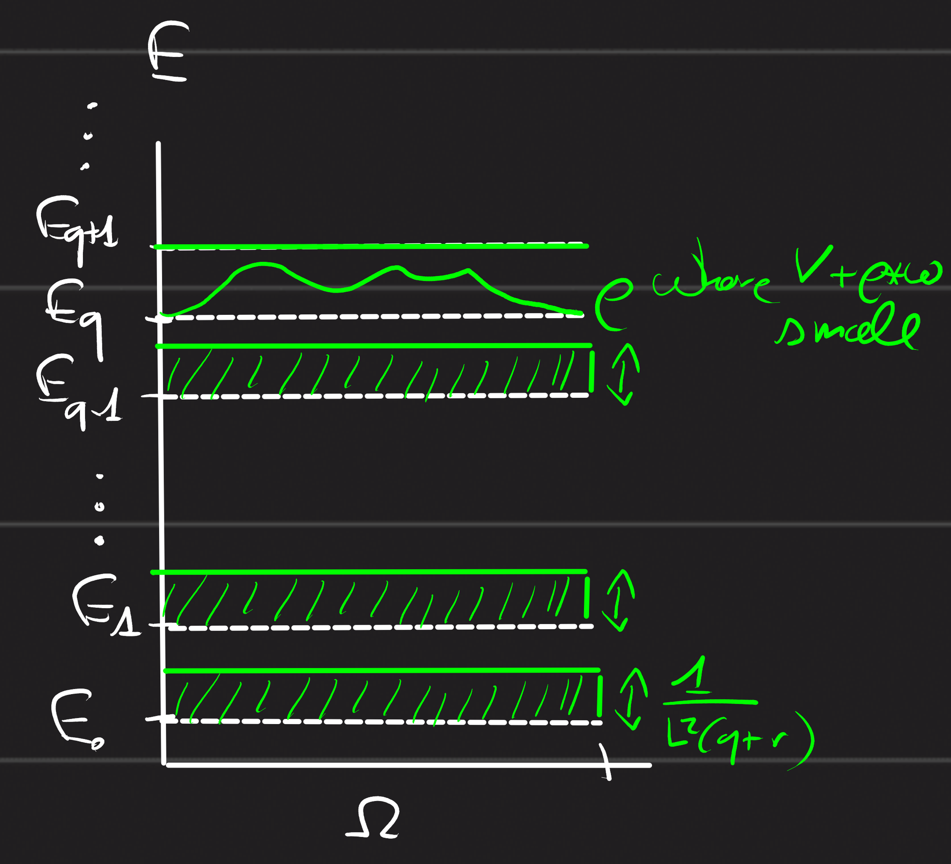
<u>reduced densities:</u> Let $\Gamma_N \in \mathfrak{L}^1(L^2_-(\Omega^N))$, $0 \leq \Gamma_N$, $\mathrm{Tr}[\Gamma_N] = 1$ (density matrix)

$\gamma_N^{(k)} = \mathrm{Tr}_{k+1:N}[\Gamma_N]$ $\quad \left(A, B \in \mathfrak{L}^1(\Omega),\ A \otimes B \in \mathfrak{L}^1(\Omega^2),\ \mathrm{Tr}_1[A \otimes B] = \mathrm{Tr}[A]\, B\right)$

$\gamma_N^{(k)}(x_{1:k}; y_{1:k}) = \int_{\Omega^{N-k}} \Gamma_N(x_{1:k}, z_{k+1:N}; y_{1:k}, z_{k+1:N})\, dz_{k+1:N}$, $\quad \rho_N^{(k)}(x_{1:k}) := \gamma_N^{(k)}(x_{1:k}, x_{1:k})$

Theorem (Convergence of reduced densities)

$$\boxed{\begin{array}{l} \text{If } \Gamma_N = |\Psi_N\rangle\langle\Psi_N| \text{ with } \Psi_N \text{ minimizing the quantum } N \text{ body energy}, \ V, \omega \in L^2(\Omega) \text{ then} \\ \exists \mu \in \mathcal{P}\left(\frac{r}{q+r}\mathcal{P}(\Omega)\right) \text{ only charging minimizers of } \mathcal{E}_{qLL} \text{ such that} \\ \text{in the sense of Radon measures, } \forall k \in \mathbb{N}^*, \quad \rho_N^{(k)} \underset{N \to +\infty}{\overset{*}{\rightharpoonup}} \int \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho) \end{array}}$$



Tools and sketch of the proof:

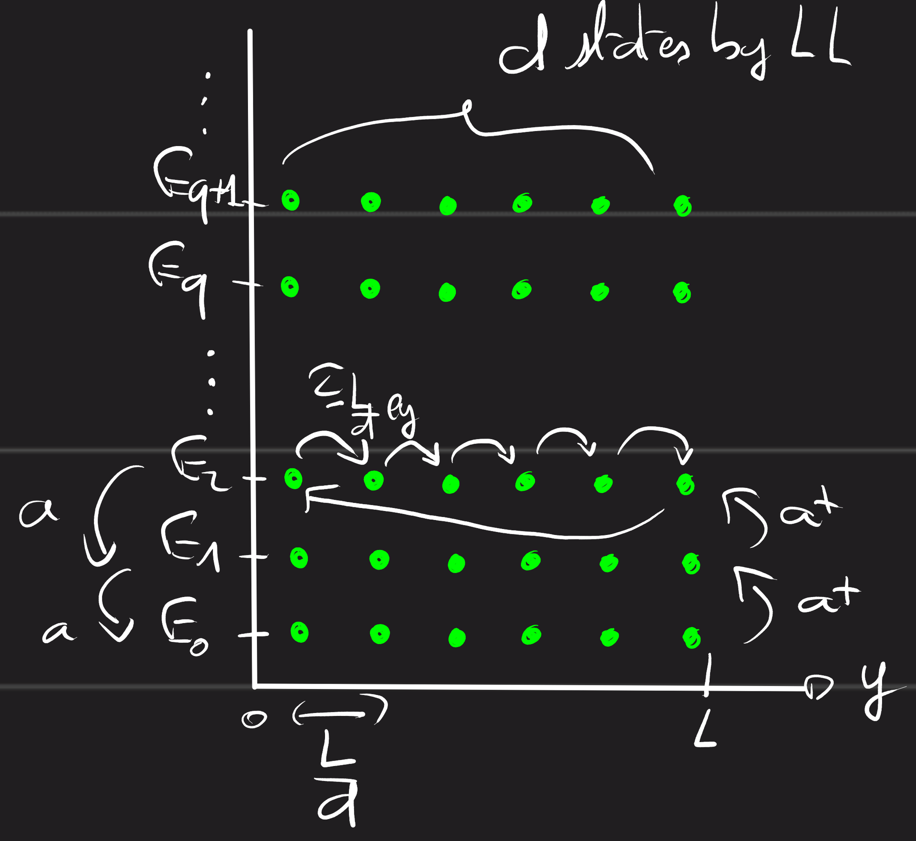
<u>quantization</u> . Landau levels: $P = i\hbar\nabla + bA =: \begin{pmatrix} \Pi_x \\ \Pi_y \end{pmatrix}$, $a := \frac{\Pi_y - i\Pi_x}{\sqrt{2\hbar b}}$, $a^+ := \frac{\Pi_y + i\Pi_x}{\sqrt{2\hbar b}} \to [a, a^+] = 1$,

$\mathcal{N} := a^+ a \to \mathcal{L} = 2\hbar b\left(\mathcal{N} + \frac{1}{2}\right)$, $n LL = \{\Psi \in \mathrm{Dom}(\mathcal{L}) \mid \mathcal{N}\Psi = n\Psi\}$, $sp(\mathcal{N}) = \mathbb{N}$

. Caracterization of $\mathrm{d}L$: $\to \mathrm{0}LL \subset \mathcal{O}(\Omega)\, e^{-\frac{\Phi}{\ell_b^2}}$

$\to f e^{-\Phi/\ell_b^2} \in \mathrm{0}LL \Rightarrow f$ has d zeros inside Ω

$\to \dim(\mathrm{0}LL) = d$ (d independant Fourier coefficient),



In Landau Gauge: $\Psi_{00}(x,y) = \frac{\pi^{-1/4}}{\sqrt{L\ell_b}} \sum_{k\in\mathbb{Z}}' e^{2i\pi k d\frac{x}{L} - \frac{1}{2\ell_b^2}(y+kL)^2}$, $\Psi_{n\ell} = \frac{a^{+n}}{\sqrt{n!}} \tau_{-\frac{L}{d}e_y}^{\ell} \Psi_{00}$

<u>projectors</u>: $\Pi_n = \sum_{\ell=0}^{d-1} |\Psi_{n\ell}\rangle\langle\Psi_{n\ell}|$, $g_x^2 \to \delta_{0_{\mathbb{R}^2}}$ and localize $\Pi_n \to \Pi_{n,x} := g_x(\cdot - x)\,\Pi_n\, g_x(\cdot - x)$,

projection on point $n,x \in \mathbb{N}\times\Omega$ on phase space

<u>prop</u>: $\Pi_n \sim \Pi_n^\infty$, $\int \Pi_x \, d\eta(x) = 1$,

<u>semi-classical limit</u> Husimi functions: $m_N^{(k)}(X_{1:k}) := \mathrm{Tr}\left[\Gamma_N \bigotimes_{i=1}^{k} \Pi_{X_i}\right]$, <u>prop</u> symmetry, $\int m^{(k)} d\eta^{\otimes k} = 1$, $0 \leq m^{(k)} \leq \frac{1}{\left(L^2(q+r)\right)^{k+o(1)}}$ (Pauli principle)

$M_N = \left(m_N^{(k)}\right)_{k\in\mathbb{N}^*}$, $\eta = \sum_{n\in\mathbb{N}}' \delta_n \otimes \mathrm{leb} \in \mathcal{M}(\mathbb{N}\times\Omega)$

$$\mathcal{E}_{sc}[M_N] := \int_{\mathbb{N}\times\Omega} E_n\, m_N^{(1)}(n,x)\, d\eta(n,x) + \int_{\mathbb{N}\times\Omega} V(x)\, m_N^{(1)}(n,x)\, d\eta(n,x) + \int_{(\mathbb{N}\times\Omega)^2} w(x-y)\, m_N^{(2)}(n,x;\tilde{n},y)\, d\eta(n,x)\, d\eta(\tilde{n},y)$$

<u>prop</u>: $\frac{E_N^0}{N} = \mathrm{Tr}[(\mathcal{L}+V)\gamma_N^{(1)}] + \mathrm{Tr}[w\gamma_N^{(2)}] = \mathcal{E}_{sc}[M_N] + o(1)$

<u>mean field limit</u> decorrelation (to justify!) $m^{(2)} = m^{(1)} \otimes m^{(1)}$

. saturation of low LL: let $\rho \in \mathcal{D}_{qLL}$ define $m_\rho(n,\cdot) = \left(\frac{1}{L^2(q+r)}\right.$ if $n<q$, ρ if $n=q$, 0 if $\left.n>q\right)$

$$\mathcal{E}_{sc}[m_\rho, m_\rho^{\otimes 2}] := \int_{\mathbb{N}\times\Omega} E_n\, m_\rho(n,x)\, d\eta(n,x) + \int_{\mathbb{N}\times\Omega} V(x)\, m_\rho(n,x)\, d\eta(n,x) + \int_{(\mathbb{N}\times\Omega)^2} w(x-y)\, m_\rho(n,x)\, m_\rho(\tilde{n},y)\, d\eta(n,x)\, d\eta(\tilde{n},y)$$

$$= \hbar b\, E^{q,r} + E_V^{q,r} + \int V\rho + E_w^{q,r} + \int V w = \hbar b\, E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}(\rho)$$

<u>upper bound</u>: Hartree-Fock theory $\to E_N^{HF} = \inf \langle\Psi_N | \mathcal{H}_N \Psi_N\rangle \overset{= \text{as that interaction}}{\geq} E_N^0$

as uncorrelated as it can get $\to \Psi_N = \frac{1}{\sqrt{N!}} \bigwedge_{i=1}^{N} \phi_i$ with $(\phi_i)_i$ orthonormal family of $L^2(\Omega)$

Wick's theorem:

$$\Psi_N = \frac{1}{\sqrt{N!}} \bigwedge_{i=1}^N \phi_i \Rightarrow \gamma_N^{(1)}(x,y) := \frac{1}{N}\sum_{i=1}^N \phi_i(x)\overline{\phi_i(y)}, \quad \gamma_N^{(2)}(x_1,x_2;y_1,y_2) = \frac{N}{N-1}\Big[\underbrace{\gamma_N^{(1)}(x_1,y_1)\gamma_N^{(1)}(x_2,y_2)}_{\text{direct}} - \underbrace{\gamma_N^{(1)}(x_1,y_2)\gamma_N^{(1)}(x_2,y_1)}_{\text{exchange},\ \gamma_{ex}}\Big]$$

Lieb's variational principle

Let $\gamma \in \mathcal{L}^1(L^2(\Omega))$, $0 \leq \gamma \leq \frac{1}{N}$, $\mathrm{Tr}[\gamma] = 1$, define γ_2 with Wick's formula then

$\exists\ \Gamma_N \in \mathcal{L}^1_+(L^2_-(\Omega^N))$, $L_2 \in \mathcal{L}^1_+(L^2(\Omega))$ such that $\gamma_N^{(1)} = \gamma$, $\gamma_N^{(2)} = \gamma_2 - L_2$

define $\gamma := \frac{L^2(q+r)}{N}\int m_\rho(X)\Pi_X\, d\mu(X)$, prop: $0 \leq \gamma \leq \frac{1}{N}$, $\mathrm{Tr}[\gamma] = 1 + o(1)$ after small modification to m_ρ so Lieb's principle applies,

$$E_N^\circ \leq \mathrm{Tr}[(\mathcal{L}+V)\gamma] + \mathrm{Tr}[\omega(\gamma_2 - L_2)] \underset{L_2 \geq 0}{\leq} \underbrace{\mathrm{Tr}[(\mathcal{L}+V)\gamma] + \mathrm{Tr}[\omega\gamma_2)}_{:= \mathcal{E}^{HF}[\gamma]} + o(1) \ \underset{\uparrow}{=}\ E_{sc}[m_\rho] + o(1)$$

control exchange terms, $\mathrm{Tr}[\gamma_{ex}] = \mathrm{Tr}[\gamma^2] \leq \frac{1}{N}$

lower bound: Theorem: De Finetti

Let $\mu \in P_s(\Omega^{\mathbb{N}})$ with marginals $(\mu^{(k)})_{k\geq 1}$, $\exists\ P_\mu \in P(P(\Omega))$ such that $\forall n \geq 1$, $\mu^{(n)} = \int_{P(\Omega)} \rho^{\otimes n}\, dP_\mu(\rho)$

after extraction:

$m_N^{(k)} \overset{*}{\rightharpoonup} m^{(k)}$ on $\sigma(L^\infty((\mathbb{N}\times\Omega)^k), L^1((\mathbb{N}\times\Omega)^k))$, $M = (m^{(k)})_k$ De Finetti: $m^{(k)} = \int_{P(\mathbb{N}\times\Omega)} m^{\otimes k}\, dP_M$

prop P_M a.e $m(n,\cdot) = \frac{1}{L^2(q+r)}$ if $n \leq q$, 0 if $n > q$, $\int m(q,\cdot)\,dx = \frac{r}{q+r}$

so $\mathcal{E}_{sc}[M_N] = \int (E_n + V(x) + \omega(x-y))\, m_N^{(2)} \rightarrow \int (E_n + V(x) + \omega(x-y))\, m^{(2)} = \int_{P(\mathbb{N}\times\Omega)} \mathcal{E}_{sc}[m]\, dP_M$

(besoin de Lieb Thirring → densité $\in L^2$)

$$= \mathrm{tot}\ E^{q,r} + E_V^{q,r} + E_\omega^{q,r} + \underbrace{\int_{P(\mathbb{N}\times\Omega)} \mathcal{E}_{QLL}[\rho]\, dP_M(m)}$$

$$= \int_{\frac{r}{q+r}P(\Omega)} \mathcal{E}_q[\rho]\, d\mu(\rho) \qquad E_N^\circ \geq \mathcal{E}_{sc}[M_N] \Rightarrow \mathcal{E}^\circ_{QLL} \geq \int_{\frac{r}{q+r}P(\Omega)} \mathcal{E}_{QLL}[\rho]\, d\mu(\rho)$$